Q Find the co-ordinates of the second focus and the equation of the second directrix of the ellipse whose one focus is $S(1,2)$ and corresponding directrix is $x-y=5$ and $e=\frac{1}{2}$.

Soln The major axis through the focus $S(1,2)$ ⊥ to the directrix $x-y=5$ → ①

$(y-2)=m(x-1)$ is the eqn of line passing through $(1,2)$.

If it is ⊥ to the line $x-y=5$

$m=-1$

Hence the eqn of major axis is

$y-2=(-1)(x-1)$

$\Rightarrow y+x=3$ → ②

Solving eqns ① & ② we get the point of intersection of major axis and directrix

① + ② $\Rightarrow 2y=-2, \Rightarrow y=-1$

Now $x+1=5 \Rightarrow x=4$

The required co-ordinates of the point X are $(4,-1)$

The vertices A_1 and A divide the line SX internally & externally in the ratio

$A:A_1=2:1$, $A_1:A=1:2$

Hence $A_1=(2,1)$, $A=(-2,1)$

The centre C is middle point of AA_1 with the co-ordinates $(0,1)$.

If $S_1(\alpha,\beta)$ be the second focus and $C(0,1)$ be the middle point of SS_1

then we already have $S = (1, 2)$

$$\Rightarrow \quad 0 = \frac{\alpha + 1}{2} \Rightarrow \alpha = -1$$

$$1 = \frac{\beta + 2}{2} \Rightarrow \beta = 0$$

Hence coordinates of $\boxed{S_1 \text{ is } (-1, 0)}$

Also, C is centre of XX_1, therefore

$X_1 = (h, k)$

$$\Rightarrow \quad 0 = \frac{h + 4}{2} \Rightarrow h = -4$$

$$1 = \frac{k - 1}{2} \Rightarrow k = 3$$

$\therefore \quad X_1 = (-4, 3)$

Now the directrix M_1X_1 is a line parallel to the given directrix $x - y = 5$. and passes through the point $X_1(-4, 3)$.

Hence its equation is

Focus is (a, b)

then eqn of directrix $\Rightarrow x + a = y \mp b$

$\Rightarrow x - 4 = y \mp 3 \Rightarrow x - y - 7 = 0$

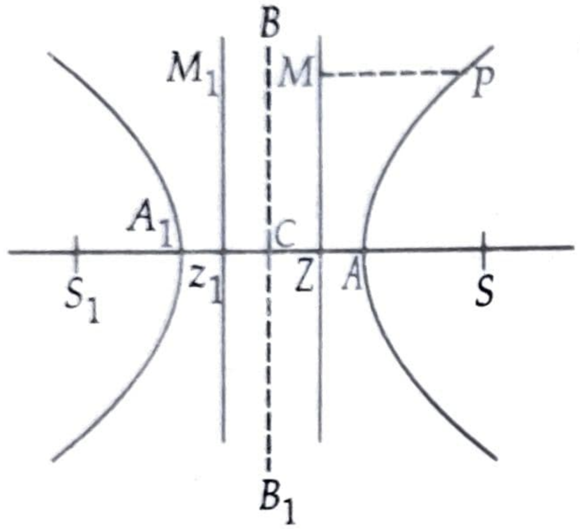
or $\boxed{x - y = 7}$

5.1. DEFINITION

A hyperbola is the locus of a point which moves such that its distance from a fixed point bears a constant ratio (>1) to its distance from a fixed line. As before, the fixed point is called the focus, the fixed line, the directrix and the constant ratio is called the eccentricity.

The eccentricity 'e' in the case of hyperbola is always > 1.

5.2. EQUATION TO HYPERBOLA



Let S be the focus and ZM the directrix. From S draw $SZ \perp ZM$ and divide SZ internally at A and externally at A_1 in the ratio $e : 1$ so that

$$\frac{AS}{AZ} = \frac{A_1S}{A_1Z} = \frac{e}{1}.$$

Then by definition, A and A_1 are points on the hyperbola.

Let $AA_1 = 2a$ and let C be the middle point of AA_1.

Then since $\quad AS = eAZ$

and $\quad A_1S = eA_1Z$

$\therefore \quad AS + A_1S = e(AZ + A_1Z)$

$\Rightarrow \quad (CS - CA) + (A_1C + CS) = e(CA - CZ + A_1C + CZ)$

$\Rightarrow \quad 2CS = eA_1A = 2ae \ \therefore \ CS = ae.$

Also, $\quad A_1S - AS = e(A_1Z - AZ)$

$\Rightarrow \quad A_1A = e\{(A_1C + CZ) - (AC - CZ)\}$

$\Rightarrow \quad 2a = e2CZ$

$\therefore \quad CZ = a/e.$

Take C as the origin and CAX as the axis of x and a line $CY \perp CAX$ as the axis of y. Therefore the co-ordinates of S are $(ae, 0)$ and the equation to the directrix is $x = a/e$.

Let $P(x, y)$ be any point on the hyperbola. Draw $PM \perp$ the directrix ZM.

Then by definition, $SP = ePM$

$$\Rightarrow \quad SP^2 = e^2 PM^2 = e^2(x - CZ)^2$$

$$\Rightarrow \quad (x - ae)^2 + y^2 = e^2(x - a/e)^2$$

$$\Rightarrow \quad x^2 + y^2 - 2aex + a^2e^2 = e^2x^2 - 2aex + a^2$$

$$\Rightarrow \quad x^2(e^2 - 1) - y^2 = a^2(e^2 - 1) \quad \because e > 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

Since $e > 1$, $a^2(e^2 - 1)$ is positive; hence writing $b^2 = a^2(e^2 - 1)$ the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(1)}$$

which is the standard equation to the hyperbola.

We have seen in the case of the ellipse that it has second focus and a second directrix.

Here also if we take a point S_1 on the x-axis, left of the origin C, such that $CS_1 = CS = ae$ and another point Z_1 such that $CZ_1 = CZ = a/e$, it can be easily shown that the equation (1) still holds true and hence S_1 is the second focus and Z_1M_1 a second directrix.

5.3. DEFINITIONS

The points A and A_1 are called the vertices of the hyperbola and C its centre.

The line AA_1 is called the *transverse axis.*

If B and B_1 be points on OY such that $OB = OB_1 = b$, then BB_1 (= $2b$) is called the *conjugate axis*.

The chord through the focus parallel to the directrix is called the *latus rectum* and its is not difficult to show that the length of the latus rectum (as in the case of ellipse) = $2b^2/a$.

5.4. FORM OF THE CURVE

(i) Since the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ contains only even powers of x and y, the curve (hyperbola) is symmetrical w.r.t. both the x-axis and the y-axis. If (x, y) be any point on the curve, then the point $(-x, -y)$ also lies on it. Therefore the centre C bisects every chord through it.

(ii) The curve cuts the x-axis in two points $x = \pm a$. This is obtained by putting $y = 0$ in the equation of the hyperbola. The points A and A_1 thus correspond to $x = a$ and $x = -a$ respectively. The curve cuts the y-axis at points whose ordinates are given by $y^2 = -b^2$, that is, they are imaginary.

Hence the hyperbola meets the y-axis in imaginary points (i.e. it does not cut it).

(iii) Since $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ i.e. $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$, it follows that for any value of x lying between $-a$ and $+a$, y is imaginary, that is, no part of the curve lies between $x = -a$ to $x = +a$. But for other values of x outside these limits, y is real.

Similarly by writing the equation in the form $x = \pm \frac{a}{b}\sqrt{y^2 + b^2}$ we observe that for all real values of y positive or negative, x has a real value.

The curve will therefore, consist of two branches (symmetrical about the co-ordinate axes) each extending to infinity in two directions as shown in the figure.

(iv) Transforming the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to polar co-ordinates, we have

$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1$$

$$\therefore \quad \frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{\cos^2\theta}{b^2}\left(\frac{b^2}{a^2} - \tan^2\theta\right) \quad \text{...(1)}$$

Therefore so long as $\tan^2\theta < b^2/a^2$, the equation (1) gives two equal and opposite values of r corresponding to any value of θ.

When $\tan^2\theta > \frac{b^2}{a^2}$, $\frac{1}{r^2}$ becomes negative, hence r is imaginary and therefore any radius drawn at an angle greater than $\tan^{-1}\frac{b}{a}$ does not meet the hyperbola in any real points.

This shows that the hyperbola lies entirely between the two lines drawn from the centre, making angles $\pm \tan^{-1}\frac{b}{a}$ with the x-axis.

When $\tan^2\theta = \frac{b^2}{a^2}$, r becomes infinite.